

Effective Quantum Dynamics of two Brownian particles

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We use the system-plus-reservoir approach to study the quantum dynamics of a bipartite continuous variable system (two generic particles). We present an extension of the traditional model of a bath of oscillators which is capable of inducing an effective coupling between the two parts of the system depending on the choice made for the spectral density of the bath. The coupling is nonlinear in the system variables and an exponential dependence on these variables is imposed in order to guarantee the translational invariance of the model if the two particles are not subject to any external potential. The reduced density operator is obtained by the functional integral method. The dynamical susceptibility of the reservoir is modelled in order to introduce, besides a characteristic frequency, a characteristic length that determines if the effective interaction potential is strong enough to induce entanglement between the particles. Our model provides a criterion of distance for identifying in which cases a common environment can induce entanglement. Three regimes are found: the short distance regime, equivalent to a bilinear system-reservoir coupling, the long distance regime in which the particles act like coupled to independent reservoirs and the intermediate regime suitable for the competition between decoherence and induced-entanglement.

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INTRODUCTION

The usual model of Brownian motion has been successfully used to describe general properties, classical or quantum mechanical, of dissipative systems with only one degree of freedom subject to arbitrary potentials [2, 3, 4, 6, 7]. Indeed, it has been extensively shown in the literature that, within the range of interest, other approaches to dealing with dissipative systems described by a single dynamical variable always furnish us with the same results as those obtained by the bath of oscillators with a properly chosen spectral function [8, 9, 10, 11]. However, in a previous paper [1] was shown that the model of independent oscillators coupled bilinearly in coordinates to the system of interest is inappropriate to dealing with dissipative system in which two independent degrees of freedom are considered. There, a generalization of the usual model was developed extending the coupling to be nonlinear in the system variables and modifying the spectral function to mimic the low frequency limit of the response function of an interacting oscillators bath. The importance of understanding the dynamics of dissipative bipartite systems arises, for example, from quantum information and computation where it is mandatory to study the interplay between the decoherence and entanglement induced by the environment.

It is our intention in this paper to investigate the quantum behavior of a system composed by two independent particles immersed in a common environment. We use the path integral approach to find the reduced density operator, in coordinate representation, of the two particle system. The propagator that controls the temporal evolution of the system is written, as usual, in terms of the influence functional that contains the effects of the

bath. In order to guarantee that each particle behave as a Brownian particle, if the other is absent, the influence functional can be expressed as a triple product where two factors are equivalent to the result one would have obtained by coupling each particle separately to a bath of noninteracting harmonic oscillators in the usual way and the third factor introduces and effective interaction between the particles mediated by the reservoir. The latter behaves as a source of quantum correlations and allows us to find, for example, entangled states for systems initially prepared in a separable state.

As an application we calculate the temporal evolution of the entanglement (logarithmic negativity) for a Gaussian two mode state. The results show a competition between decoherence and induced entanglement that mainly depends on the temperature and the average distance between the particles.

MODEL AND EXACT DENSITY OPERATOR

In a previous paper [1], we presented a generalized system-reservoir model suitable for the study of two uncoupled particles interacting with a common bath. There, it was demonstrated that the traditional bilinear coupling is not appropriate to handle the effective interaction mediated by the bath and the solution was to introduce a nonlinear coupling in the system variables. The principal features of the model will be reviewed in this section.

Model

The Hamiltonian for the complete system is given by

One Particle Density Operator

$$H_S + H_R + H_I. \quad (1)$$

H_S is the Hamiltonian of the system of interest, H_R is the Hamiltonian of the reservoir, which will be described as a symmetrized collection of independent harmonic modes

$$H_R = \sum_{k=1}^N \left[\frac{p_k p_{-k}}{2m_k} + \frac{1}{2} m_k \omega_k^2 R_k R_{-k} \right], \quad (2)$$

and H_I is the interaction Hamiltonian which can be written in two equivalent forms [2, 3]; a coordinate-velocity coupling or a coordinate-coordinate coupling plus a new quadratic term which is necessary to preserve the translational invariance of (1) when the system of interest is not acted by any external force. The two forms are linked by a simple canonical transformation ($P_k \rightarrow m_k \omega_k R_k$ and $R_k \rightarrow \frac{P_k}{m_k \omega_k}$).

In our generalization of the previous model, the system of interest with a single degree of freedom will be represented by the free particle Hamiltonian

$$H_S = \frac{P^2}{2M}. \quad (3)$$

For the coupling term we assume the interaction Hamiltonian

$$\begin{aligned} H_I &= \frac{1}{2} \sum_{k=1}^N (C_{-k}(x)R_k + C_k(x)R_{-k}) \\ &+ \sum_{k=1}^N \frac{C_k(x)C_{-k}(x)}{2m_k \omega_k^2}. \end{aligned} \quad (4)$$

In order to represent the effect of a local interaction of the particle with a spatially homogeneous environment we choose

$$C_k(x) = \kappa_k e^{ikx}. \quad (5)$$

With this choice it is straightforward to show that the entire system is translationally invariant when there is not an external potential [1]. As a consequence of (5) the potential renormalization in (4) is a constant and therefore does not contribute to the particle dynamics. Actually, this reflects the translational invariance of whole system. A coupling like (5) appears, for example, when one deals with the interaction of a particle with the density operator of a fermionic bath [10] or in the traditional polaron problem[9].

The classical equations of motion and the implications of the coupling (5) were studied in [1] for one and two particles. There the nonlocal influence of the bath appears only when a second particle is present in the environment. Here we show that the nonlocal effects are also important for a single particle when the quantum behavior is studied but the local form[2] is recovered in a suitable limit.

In this section we use the functional integral method developed by Feynman and Vernon [5] in order to find the reduced density operator of the system of interest. The time evolution of the total density operator is given by

$$\rho(t) = \exp(-iHt/\hbar) \rho(0) \exp(iHt/\hbar), \quad (6)$$

and in coordinate representation we have

$$\begin{aligned} \langle x, \mathbf{R} | \rho(t) | y, \mathbf{Q} \rangle &= \iiint dx' dy' d\mathbf{R}' d\mathbf{Q}' \mathbf{K}(x, \mathbf{R}, t; x', \mathbf{R}', 0) \\ &\quad * \langle x', \mathbf{R}' | \rho(0) | y', \mathbf{Q}' \rangle \mathbf{K}^*(y, \mathbf{Q}, t; y', \mathbf{Q}', 0), \end{aligned} \quad (7)$$

where \mathbf{R} is a N -dimensional vector representing each degree of freedom of the bath and \mathbf{K} is the coordinate representation of the time evolution operator. The reduced density operator emerges when we eliminate the bath variables. In the coordinate representation this procedure corresponds to doing $\mathbf{R} = \mathbf{Q}$ in (7) and to integrate over all possible values of \mathbf{R} . The reduced density operator at time t depends on the total density operator at $t = 0$ and for simplicity we assume a separable initial condition

$$\rho(0) = \tilde{\rho}(0) \rho_R(0), \quad (8)$$

where $\tilde{\rho}(0)$ is the initial particle density operator and $\rho_R(0) = Z_R^{-1} \exp(-\beta H_R)$ is the bath density operator, which is assumed to be in thermal equilibrium, before the perturbation is switched on. With this assumptions we can write

$$\tilde{\rho}(x, y, t) = \iint dx' dy' J(x, y, t; x', y', 0) \tilde{\rho}(x', y', 0), \quad (9)$$

where $J(x, y, t; x', y', 0)$ is the superpropagator, which controls the time evolution of the reduced density operator and can be written as a path integral

$$\begin{aligned} J(x, y, t; x', y', 0) &= \int_{x'}^x \mathcal{D}x(t') \int_{y'}^y \mathcal{D}y(t') * \\ &\quad \exp \frac{i}{\hbar} \{ S_0[x(t')] - S_0[y(t')] \} \mathcal{F}[x(t'), y(t')]. \end{aligned} \quad (10)$$

S_0 is the action of the isolated particle and $\mathcal{F}[x(t'), y(t')]$ is a functional of the particle trajectory, which contains all the bath information [5]. The influence functional has a simple formal expression

$$\mathcal{F}[x(t'), y(t')] = \text{Tr}_R \left(\rho_R U_{RI}^\dagger[y(t')] U_{RI}[x(t')] \right), \quad (11)$$

where $U_{RI}[x(t')]$ is the unitary time evolution operator of the reservoir subjected to the influence of the system

which evolves through the Hamiltonian $H_{RI} = H_R + H_I[x(t')]$. This means that a given trajectory $x(t')$ of the system for $0 \leq t' \leq t$ acts as a forcing term to the environment. The time evolution of the operator $U_{RI}[x]$ is described by the Schrödinger equation

$$i\hbar \frac{d}{dt} U_{RI}(t) = H_{RI}(t)U_{RI}(t), \quad (12)$$

with the initial condition $U_{RI}(0) = 1$ and has the formal solution

$$U_{RI}(t) = T e^{-i \int_0^t dt' H_{RI}(t')/\hbar}, \quad (13)$$

where T is the time ordering operator. Switching to the interaction picture [13] this result can be written as

$$U_{RI}(t) = e^{-iH_R t/\hbar} T e^{-i \int_0^t dt' \tilde{H}_I[x(t')]/\hbar}, \quad (14)$$

where $\tilde{H}_I[x(t')] = e^{iH_R t/\hbar} H_I[x(t')] e^{-iH_R t/\hbar}$. Inserting (14) into (11) we have $\mathcal{F}[x(t'), y(t')] = \text{Tr}_R \left(\rho_R T e^{i \int_0^t dt' \tilde{H}_I[y(t')]/\hbar} T e^{-i \int_0^t dt' \tilde{H}_I[x(t')]/\hbar} \right)$.

Since the reservoir have many degrees of freedom we can assume that the particle induces a weak perturbation in the environment and expand the chronological product to second order in \tilde{H}_I [10],

$$\begin{aligned} T e^{-i \int_0^t dt' \tilde{H}_I[x(t')]/\hbar} &\approx 1 - \frac{i}{\hbar} \int_0^t dt' \tilde{H}_I[x(t')] \\ &\quad - \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \tilde{H}_I[x(t')] \tilde{H}_I[x(s)]. \end{aligned} \quad (15)$$

With these assumptions and tracing the reservoir variables we find

$$\begin{aligned} \mathcal{F}[x(t'), y(t')] &\approx 1 - \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \left(\langle \tilde{H}_I[x(t')] \tilde{H}_I[x(s)] \rangle + \langle \tilde{H}_I[y(s)] \tilde{H}_I[y(t')] \rangle \right. \\ &\quad \left. - \langle \tilde{H}_I[y(t')] \tilde{H}_I[x(s)] \rangle - \langle \tilde{H}_I[y(s)] \tilde{H}_I[x(t')] \rangle \right). \end{aligned}$$

Once we consider only terms to second order in $\tilde{H}_I[x(t')]$ the influence functional above is equivalent to

$$\begin{aligned} \mathcal{F}[x(t'), y(t')] &= \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \left(\langle \tilde{H}_I[x(t')] \tilde{H}_I[x(s)] \rangle + \langle \tilde{H}_I[y(s)] \tilde{H}_I[y(t')] \rangle \right. \right. \\ &\quad \left. \left. - \langle \tilde{H}_I[y(t')] \tilde{H}_I[x(s)] \rangle - \langle \tilde{H}_I[y(s)] \tilde{H}_I[x(t')] \rangle \right) \right\}. \end{aligned} \quad (16)$$

Now we calculate the averages in (16) considering the coupling (5) and invoking the reservoir translational invariance. Since we have $\langle R_k(t') R_{k'}(s) \rangle = 0$ unless $k' = -k$, one is allowed to write

$$\begin{aligned} \langle \tilde{H}_I[x(t')] \tilde{H}_I[x(s)] \rangle &= \frac{1}{2} \sum_k \{C_{-k}[x(t')] C_k[x(s)] \\ &\quad + C_k[x(t')] C_{-k}[x(s)]\} \langle R_k(t') R_{-k}(s) \rangle. \end{aligned} \quad (17)$$

The expression $\langle R_k(t') R_{-k}(s) \rangle$ can be written in terms

of the bath linear response through the fluctuation-dissipation theorem. We call for simplicity $\alpha_k(t' - s) = \langle R_k(t') R_{-k}(s) \rangle$ and

$$\alpha_k(t' - s) = \frac{\hbar}{\pi} \int_{-\infty}^{\infty} d\omega \text{Im} \tilde{\chi}_k(\omega) \frac{e^{-i\omega(t'-s)}}{1 - e^{-\omega\hbar\beta}}, \quad (18)$$

where $\tilde{\chi}_k(\omega)$ is the dynamical susceptibility of the reservoir. So, the averages in (16) have the final form

$$\langle \tilde{H}_I[x(t')] \tilde{H}_I[x(s)] \rangle = \frac{1}{2} \sum_k \{C_{-k}[x(t')] C_k[x(s)] + C_k[x(t')] C_{-k}[x(s)]\} \alpha_k(t' - s), \quad (19)$$

and similar ones for the other terms. Putting all those together we find the following nonlinear influence functional

$$\begin{aligned} \mathcal{F}[x(t'), y(t')] &= \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \sum_k [\kappa_k \kappa_{-k} (\cos k[x(t') - x(s)] - \cos k[y(t') - x(s)]) \alpha_k(t' - s) \right. \\ &\quad \left. + \kappa_k \kappa_{-k} (\cos k[y(t') - y(s)] - \cos k[y(s) - x(t')]) \alpha_k^\dagger(t' - s)] \right\}. \end{aligned} \quad (20)$$

The function $\alpha_k(t' - s)$ is related to the bath linear response by the Fourier transform (18). At this point it is necessary to model the bath dynamical susceptibility $\tilde{\chi}_k(\omega)$, since it is not our intention to completely describe the microscopic details of the bath. We can assume the imaginary part of the bath dynamical susceptibility has the form

$$\text{Im}\tilde{\chi}_k(\omega) = f(k)\omega\theta(\omega - \Omega). \quad (21)$$

Here we introduce a high frequency cutoff Ω as the characteristic frequency of the bath. The Markov dynamics is achieved when we take the limit $\Omega \rightarrow \infty$, and the function $f(k)$ responds for the nonlocal influence of the bath. This approximation is equivalent to replace the free oscillator response function by the low frequency limit of the damped oscillator response function [1] and allow us to separate the characteristic time and length scales of the reservoir. A functional dependence like (21) for the dynamical response of the bath has been employed in Refs. [10, 11] for fermionic environments.

The parity of $\text{Im}\tilde{\chi}_k(\omega)$ allow us to write the correlation

function $\alpha_k(t' - s)$ as

$$\alpha_k(t' - s) = \alpha_k^{(R)}(t' - s) + i\alpha_k^{(I)}(t' - s), \quad (22)$$

with the real and imaginary parts defined respectively as

$$\begin{aligned} \alpha_k^{(R)}(t' - s) &= \\ \frac{\hbar}{\pi} \int_0^\infty d\omega \text{Im}\tilde{\chi}_k(\omega) \cos \omega(t' - s) \coth(\hbar\beta\omega/2) \end{aligned} \quad (23)$$

and

$$\alpha_k^{(I)}(t' - s) = -\frac{\hbar}{\pi} \int_0^\infty d\omega \text{Im}\tilde{\chi}_k(\omega) \sin \omega(t' - s). \quad (24)$$

The functional (20) can also be written in terms of these real and imaginary parts. With prescription (21) we can evaluate the frequency integrals in the imaginary part (24) taking the limit $\Omega \rightarrow \infty$ and considering that the coupling was switched on at $t = 0^+$. Following this procedure we have a functional with a Markovian imaginary part and a explicit nonlinear dependence,

$$\begin{aligned} \mathcal{F}[x(t'), y(t')] = \exp \left\{ \frac{i}{2\hbar} \sum_k \kappa_k \kappa_{-k} f(k) k \int_0^t dt' \sin k[y(t') - x(t')] (\dot{x}(t') + \dot{y}(t')) \right. \\ - \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \sum_k \kappa_k \kappa_{-k} [\cos k[x(t') - x(s)] - \cos k[y(t') - x(s)] \\ \left. + \cos k[y(t') - y(s)] - \cos k[y(s) - x(t')]] \alpha_k^{(R)}(t' - s) \right\}. \end{aligned}$$

This functional can be approximated assuming that the most important trajectories $(x(t'), y(t'))$ are confined within a region small compared to a characteristic length k_0^{-1} introduced in (21) through the function $f(k)$. For

example, in fermionic environments this length is related to the Fermi wave number k_F [10, 11]. In this approximation, we have $k(y(t') - x(t')) \ll 1$ and the functional can be written as

$$\begin{aligned} \mathcal{F}[x(t'), y(t')] = \exp \left\{ \frac{i\eta}{2\hbar} \int_0^t dt' (y(t') - x(t')) (\dot{x}(t') + \dot{y}(t')) \right. \\ \left. - \frac{\eta}{\hbar\pi} \int_0^t dt' \int_0^{t'} ds \int_0^\infty d\omega \omega \coth \left(\frac{\hbar\beta\omega}{2} \right) (x(t') - y(t')) \cos \omega(t' - s) (x(s) - y(s)) \right\}, \end{aligned} \quad (25)$$

where we have identified $\eta = \sum_k k^2 \kappa_k \kappa_{-k} f(k)$. Notice that, with this modification, we obtain a relation between the damping constant and some microscopic parameters of the oscillator bath. The functional (25) coincides with

the result obtained by coupling the particle of interest bilinearly to a bath of noninteracting harmonic oscillators with the spectral function $J(\omega) = \eta\omega$ [2, 3].

Two Particle Density Operator

Now we are going to study the dynamics of a system with two degrees of freedom immersed in a dissipative environment. In this case the Lagrangian of the system of interest is

$$L_S = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2, \quad (26)$$

and the coupling term

$$\begin{aligned} L_I = -\frac{1}{2} \sum_k & [(C_{-k}(x_1) + C_{-k}(x_2)) R_k \\ & + (C_k(x_1) + C_k(x_2)) R_{-k}]. \end{aligned} \quad (27)$$

Notice that we have not included any counter-term in (27) since our system is manifestly translationally invariant. The time evolution of the reduced density operator for the two particle system is given by

$$\tilde{\rho}(\mathbf{x}, \mathbf{y}, t) = \iint d^2x d^2y J(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', 0) \tilde{\rho}(\mathbf{x}', \mathbf{y}', 0), \quad (28)$$

and the superpropagator for this case is

$$\begin{aligned} J(\mathbf{x}, \mathbf{y}, t; \mathbf{x}', \mathbf{y}', 0) = & \int_{\mathbf{x}'}^{\mathbf{x}} \mathcal{D}\mathbf{x}(t') \int_{\mathbf{y}'}^{\mathbf{y}} \mathcal{D}\mathbf{y}(t') \\ & \exp \frac{i}{\hbar} \{ S_0[\mathbf{x}(t')] - S_0[\mathbf{y}(t')] \} F[\mathbf{x}(t'), \mathbf{y}(t')], \end{aligned} \quad (29)$$

where we have defined the vectors $\mathbf{x}(t') = (x_1(t'), x_2(t'))$ and $\mathbf{y}(t') = (y_1(t'), y_2(t'))$. S_0 is the action of the isolated two particle system and F is the Feynman-Vernon influence functional, which in the operator form can be written as

$$F[\mathbf{x}(t'), \mathbf{y}(t')] = \text{Tr}_R \left(\rho_R U_{RI}^\dagger[\mathbf{y}(t')] U_{RI}[\mathbf{x}(t')] \right), \quad (30)$$

where $U_{RI}[\mathbf{x}(t')]$ is the unitary time evolution operator as in (13). The procedure to calculate the influence functional for the two particle case is completely equivalent

to the one particle case and we can directly generalize the result, keeping only terms to second order in $H_I[\mathbf{x}(t')]$, as

$$\begin{aligned} F[\mathbf{x}(t'), \mathbf{y}(t')] = & \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \right. \\ & \times \left(\langle \tilde{H}_I[\mathbf{x}(t')] \tilde{H}_I[\mathbf{x}(s)] \rangle + \langle \tilde{H}_I[\mathbf{y}(s)] \tilde{H}_I[\mathbf{y}(t')] \rangle \right. \quad (31) \\ & \left. \left. - \langle \tilde{H}_I[\mathbf{y}(t')] \tilde{H}_I[\mathbf{x}(s)] \rangle - \langle \tilde{H}_I[\mathbf{y}(s)] \tilde{H}_I[\mathbf{x}(t')] \rangle \right) \right\}. \end{aligned}$$

Again, we take the average over the equilibrium density matrix of the bath and using the fluctuation dissipation theorem and the correlation function defined in (22), we can write the two particle influence functional as

$$\begin{aligned} F[\mathbf{x}(t'), \mathbf{y}(t')] = & \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \tilde{K}[\mathbf{x}, \mathbf{y}, t', s] \alpha_k^{(R)}(t' - s) \right. \\ & \left. - \frac{i}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \tilde{K}[\mathbf{x}, \mathbf{y}, t', s] \alpha_k^{(I)}(t' - s) \right\}, \end{aligned} \quad (33)$$

where we have defined the nonlinear kernel

$$\begin{aligned} \tilde{K}[\mathbf{x}, \mathbf{y}, t', s] = & \sum_k \kappa_k \kappa_{-k} \sum_{i,j=1}^2 [\cos k[x_i(t') - x_j(s)] + \cos k[y_i(t') - y_j(s)] \\ & - \cos k[y_i(t') - x_j(s)] - \cos k[x_i(t') - y_j(s)]]. \end{aligned} \quad (34)$$

The imaginary part of (33) can be reduced using the response function (21) in a form completely analogous to the one particle case. The resulting imaginary part of the exponent of the influence functional can be written, in a compact form, as

$$-\frac{1}{2\hbar} \int_0^t dt' \tilde{L}[\mathbf{x}, \mathbf{y}, t'] + \frac{2\Omega}{\hbar\pi} \int_0^t dt' V[\mathbf{x}, \mathbf{y}, t'], \quad (35)$$

where we have defined the instantaneous kernels

$$\begin{aligned} \tilde{L}[\mathbf{x}, \mathbf{y}, t'] = & \sum_k \kappa_k \kappa_{-k} k f(k) \left(\sum_{i,j=1}^2 \sin k[x_i(t') - y_j(t')] [\dot{x}_i(t') + \dot{y}_j(t')] \right. \\ & \left. + \sin k[x_1(t') - x_2(t')] [\dot{x}_2(t') - \dot{x}_1(t')] - \sin k[y_1(t') - y_2(t')] [\dot{y}_2(t') - \dot{y}_1(t')] \right) \end{aligned} \quad (36)$$

and

$$V[\mathbf{x}, \mathbf{y}, t'] = \sum_k \kappa_k \kappa_{-k} f(k) (\cos k[x_1(t') - x_2(t')] - \cos k[y_1(t') - y_2(t')]). \quad (37)$$

The well-known effects of dissipation and diffusion are included in the kernel $\tilde{L}[\mathbf{x}, \mathbf{y}, t']$. Note that this term contains direct and indirect influence of dissipation. On the other hand, the function $V[\mathbf{x}, \mathbf{y}, t']$ introduces a completely new effect that can be interpreted as an effective interaction mediated by the environment. The functional at this point can be written as

$$\begin{aligned} F[\mathbf{x}(t'), \mathbf{y}(t')] = & \\ & \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \tilde{K}[\mathbf{x}, \mathbf{y}, t', s] \alpha_k^{(R)}(t' - s) \right. \\ & \left. - \frac{i}{2\hbar} \int_0^t dt' \tilde{L}[\mathbf{x}, \mathbf{y}, t'] + \frac{i2\Omega}{\hbar\pi} \int_0^t dt' V[\mathbf{x}, \mathbf{y}, t'] \right\}. \quad (38) \end{aligned}$$

The functional above is still too complex and additional assumptions are necessary. Since we are concerned only with the effective terms mediated by the reservoir, we can assume that each particle trajectory is localized within a region restricted by the characteristic length of the reservoir. That leads to the approximation $k[x_i(t') - y_i(s)] \ll 1$ and taking only the terms up to second order in $k[x_i(t') - y_i(s)]$ the functional can be written as

$$\begin{aligned} F[\mathbf{x}(t'), \mathbf{y}(t')] = & \mathcal{F}[x_1(t'), y_1(t')] \mathcal{F}[x_2(t'), y_2(t')] \\ & \exp \left\{ -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} ds \bar{K}[\mathbf{x}, \mathbf{y}, t', s] \alpha_k^{(R)}(t' - s) \right. \\ & \left. - \frac{i}{2\hbar} \int_0^t dt' \bar{L}[\mathbf{x}, \mathbf{y}, t'] + \frac{i2\Omega}{\hbar\pi} \int_0^t dt' V[\mathbf{x}, \mathbf{y}, t'] \right\}, \quad (39) \end{aligned}$$

where \bar{K} and \bar{L} are defined as in (34) and (36), respectively, but excluding the terms with $i = j$. The factor $\mathcal{F}[x_i(t'), y_i(t')]$ was defined in (25) and corresponds to the influence functional within the local approximation of the Brownian particle dynamics. The functional above is formed by three clearly distinct parts, the first two correspond to the direct influence of the bath over each particle separately and the third part contains the interaction between the particles mediated by the bath.

Towards a better understanding of the interaction factor in (39) it is important to study the short and long distance limits, which are obviously defined in relation to the characteristic length of the reservoir k_0^{-1} . For the first we assume $k[x_i(t') - x_j(s)] \ll 1$ for all particle trajectories and for all times, which means that the particles are very close and the dynamics is essentially local. In this approximation the superpropagator reduces to

$$\begin{aligned} J = & \exp \left\{ \frac{-i}{\hbar} h(X, Y) \right\} \int_{\mathbf{x}'}^{\mathbf{x}} \int_{\mathbf{y}}^{\mathbf{y}} \mathcal{D}\mathbf{x}(t') \mathcal{D}\mathbf{y}(t') \exp \frac{i}{\hbar} \left\{ S_0[\mathbf{x}(t')] - S_0[\mathbf{y}(t')] - \frac{\eta}{2} \int_0^t (x_1 \dot{y}_1 - y_1 \dot{x}_1) dt' - \frac{\eta}{2} \int_0^t (x_2 \dot{y}_2 - y_2 \dot{x}_2) dt' \right. \\ & - \frac{\eta}{2} \int_0^t dt' [x_1 \dot{y}_2 - y_2 \dot{x}_1 + x_2 \dot{y}_1 - y_1 \dot{x}_2] - \frac{\eta\Omega}{\pi} \int_0^t dt' \left[(x_1 - x_2)^2 - (y_1 - y_2)^2 \right] \left. \right\} \\ & \times \exp \left\{ -\frac{\eta}{\hbar\pi} \int_0^t dt' \int_0^{t'} ds \sum_{i,j=1}^2 [x_i(t') - y_i(t')] K(t' - s) [x_j(s) - y_j(s)] \right\}, \end{aligned}$$

where we have introduced the temperature dependent kernel

$$K(t' - s) = \int_0^\Omega d\omega \omega \coth(\hbar\beta\omega/2) \cos\omega(t' - s) \quad (40)$$

and the function

$$\begin{aligned} h(X, Y) = & \frac{\eta}{4} [(x_1 + x_2)^2 - (x'_1 + x'_2)^2 \\ & - (y_1 + y_2)^2 + (y'_1 + y'_2)^2]. \quad (41) \end{aligned}$$

Now, for simplicity, we can use two successive changes of variables. The first is defined as $q_i(t) = (x_i(t) + y_i(t))/2$, $\xi_i(t) = x_i(t) - y_i(t)$ and can be interpreted as the center and width of the wave packets respectively. The second introduces the center of mass and relative coordinate variables defined respectively as $r(t) = (q_1(t) + q_2(t))/2$, $u(t) = q_1(t) - q_2(t)$ and the auxiliar variables $\chi(t) = (\xi_1(t) + \xi_2(t))/2$, $v(t) = \xi_1(t) - \xi_2(t)$. With these replacements the local version of the superpropagator reads

$$\begin{aligned}
J = & \exp \left\{ \frac{-i}{\hbar} \tilde{h}(r, \chi) \right\} \int_{r'}^r \int_{\chi'}^{\chi} \int_{u'}^u \int_{v'}^v \mathcal{D}r(t') \mathcal{D}\chi(t') \mathcal{D}u(t') \mathcal{D}v(t') \exp \left\{ -\frac{4\eta}{\hbar\pi} \int_0^t dt' \int_0^{t'} ds K(t' - s) \chi(t') \chi(s) \right\} \\
& \exp \frac{i}{\hbar} \left\{ \Sigma[r, \chi, u, v] - 2\eta \int_0^t dt' [\chi(t') \dot{r}(t') - r(t') \dot{\chi}(t')] - \frac{2\eta\Omega}{\pi} \int_0^t dt' u(t') v(t') dt' \right\}, \tag{42}
\end{aligned}$$

with

$$\Sigma[r, \chi, u, v] = \int_0^t M(2\dot{r}\dot{\chi} + \dot{u}\dot{v}/2) dt' \tag{43}$$

and

$$\tilde{h}(r, \chi) = 2\eta(r\chi - r'\chi'). \tag{44}$$

From the local functional (42) we can point out the following features. Firstly, the stationary trajectory of $r(t)$, deduced from the action functional in the propagator's exponent, describes the dissipative dynamics of a particle with mass $2M$, that is the center of mass of the two Brownian particle system. The second, and more remarkable result is that, the relative coordinate $u(t)$ in

this approximation describes a dissipationless dynamics of a variable acted upon by an effective force which induces an interaction between the individual parts of the system of interest. This effect had previously been noticed in a classical approach and coincides with that of other works in which the quantum evolution was studied using the master equation approach and a bilinear system-bath coupling [15, 16].

In the long distance limit we suppose that each particle is restricted to move within separated regions. In this case the distance L between the regions is considered bigger than the characteristic length of the reservoir and it is possible to approximate $k[x_i(t') - y_i(s)] \gg 1$. When this condition is satisfied the interaction terms are negligible and the superpropagator reads

$$\begin{aligned}
J = & \int_{r'}^r \int_{u'}^u \int_{\chi'}^{\chi} \int_{v'}^v \mathcal{D}r(t') \mathcal{D}u(t') \mathcal{D}\chi(t') \mathcal{D}v(t') \exp \left\{ \frac{i}{\hbar} \int_0^t dt' M \left(\frac{\dot{u}(t')\dot{v}(t')}{2} + 2\dot{r}(t')\dot{\chi}(t') \right) \right\} \\
& \exp \left\{ -\frac{i\eta}{\hbar} \int_0^t dt' \left(\frac{\dot{u}(t')v(t')}{2} + 2\dot{r}(t')\chi(t') \right) - \frac{\eta}{\hbar\pi} \int_0^t dt' \int_0^{t'} ds \left(\frac{v(s)v(t')}{2} + 2\chi(s)\chi(t') \right) K(t' - s) \right\}. \tag{45}
\end{aligned}$$

It is obvious that in the long distance limit the dynamics is completely equivalent to the case of two uncoupled particles interacting with two independent but identical environments.

Now, in order to study an approach suitable for interparticle distances between the two above-mentioned limits, it is necessary to introduce new assumptions. We consider the variables insensitive to the interparticle distance negligible in relation to the characteristic length k_0^{-1} , i. e. the variables v and χ , which are related to the

width of the wave packets, can be kept as very small when compared with k and the approximations $\cos[k\chi(t')] \approx 1$, $\sin[k\chi(t')] \approx k\chi(t')$, $\sin[kv(t')/2] \approx kv(t')/2$ appear to be adequate ones. On the other hand, the nonlinear terms involving the variables r and u will be replaced by a phenomenological function parametrized by the relation between the characteristic length of the bath and the average distance between the physically allowed regions for the particle trajectories. In this way the superpropagator can be written as

$$\begin{aligned}
J = & \exp \left\{ \frac{i}{\hbar} \tilde{h}(r, \chi, u, v) \right\} \int_{r'}^r \int_{u'}^u \int_{\chi'}^{\chi} \int_{v'}^v \mathcal{D}r(t') \mathcal{D}u(t') \mathcal{D}\chi(t') \mathcal{D}v(t') \\
& \times \exp \frac{i}{\hbar} \left\{ M \int_0^t dt' (2\dot{r}\dot{\chi} + \dot{u}\dot{v}/2) + \frac{\eta}{4} \int_0^t dt' (4\dot{\chi}(t')r(t') + \dot{v}(t')u(t') - \dot{u}(t')v(t') - 4\dot{r}(t')\chi(t')) \right. \\
& \quad \left. - \frac{\eta}{2} \int_0^t dt' (D(k_0 L)\dot{v}(t')u(t') + 4D(k_0 L)\dot{r}(t')\chi(t')) - \frac{2\eta\Omega}{\pi} \int_0^t dt' D(k_0 L)u(t')v(t') \right\} \\
& \times \exp \left\{ -\frac{\eta}{\hbar\pi} \int_0^t dt' \int_0^{t'} ds K(t' - s) \left\{ 2(1 + D(k_0 L))\chi(t')\chi(s) + \frac{1}{2}(1 - D(k_0 L))v(t')v(s) \right\} \right\}. \quad (46)
\end{aligned}$$

where we redefine

$$\begin{aligned}
\tilde{h}(r, \chi, u, v) = & \frac{\eta}{4} (4r'\chi' - 4r\chi + u'v' - uv) \\
& - \frac{\eta}{2} D(k_0 L) (u'v' - uv), \quad (47)
\end{aligned}$$

and introduce the function

$$D(k_0 L) = \exp(-k_0 L), \quad (48)$$

with the parameter L being a definition of the average length between the regions where the particles move. Actually one could use it as the initial distance between the centers of the packets once one can make sure they will not overlap in the long run.

The functional integrals in (46) can be evaluated directly since all the variables appear to second order only. The result of this integration is

$$\begin{aligned}
J = & \mathcal{N}(t) \exp \frac{i}{\hbar} \left\{ \tilde{h}(r, \chi, u, v) - \frac{M}{\sinh[t\gamma_+/2]} \left\{ \gamma_+ (r' - r) \left(\chi e^{-\gamma_+ t/2} - \chi' e^{\gamma_+ t/2} \right) - \gamma (r\chi - r'\chi') \sinh[t\gamma_+/2] \right\} \right. \\
& - \frac{M}{4 \sinh \left[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right]} \left\{ \sqrt{\gamma_-^2 - 4\omega_t^2} \left(u'v e^{-\gamma_- t/2} + uv' e^{\gamma_- t/2} \right) \right. \\
& \quad \left. \left. - (\gamma_- - \gamma) (uv - u'v') \sinh \left[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right] - \sqrt{\gamma_-^2 - 4\omega_t^2} (uv + u'v') \cosh \left[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right] \right\} \right\} \\
& \times \exp -\frac{1}{\hbar} \left\{ (1 + D(k_0 L)) (A_\chi(t)\chi^2 + B_\chi(t)\chi\chi' + C_\chi(t)\chi'^2) + (1 - D(k_0 L)) (A_v(t)v^2 + B_v(t)vv' + C_v(t)v'^2) \right\}. \quad (49)
\end{aligned}$$

In this expression we use the definitions $\omega_0^2 = \frac{4\Omega\eta}{M\pi}$, $\gamma_\pm = \gamma(1 \pm D[k_0 L])$, $\omega_t^2 = \omega_0^2 D(k_0 L)$ and the time dependent functions are

$$A_\chi(t) = \frac{\eta}{\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) e^{-\gamma_+ t} \frac{\sinh[s\gamma_+/2] \sinh[\tau\gamma_+/2]}{\sinh[t\gamma_+/2]^2} e^{\gamma_+(s+\tau)/2} \quad (50)$$

$$B_\chi(t) = \frac{2\eta}{\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) e^{-\gamma_+ t/2} \frac{\sinh[\tau\gamma_+/2] \sinh[(t-s)\gamma_+/2]}{\sinh[t\gamma_+/2]^2} e^{\gamma_+(s+\tau)/2} \quad (51)$$

$$C_\chi(t) = \frac{\eta}{\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) \frac{\sinh[(t-s)\gamma_+/2] \sinh[(t-\tau)\gamma_+/2]}{\sinh[t\gamma_+/2]^2} e^{\gamma_+(s+\tau)/2}, \quad (52)$$

$$A_v(t) = \frac{\eta}{4\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) e^{-\gamma_- t} \frac{\sinh[\frac{s}{2} \sqrt{\gamma_-^2 - 4\omega_t^2}] \sinh[\frac{\tau}{2} \sqrt{\gamma_-^2 - 4\omega_t^2}]}{\sinh[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2}]^2} e^{\gamma_-(s+\tau)/2} \quad (53)$$

$$B_v(t) = \frac{\eta}{2\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) e^{-\gamma_- t/2} \frac{\sinh \left[\frac{\tau}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right] \sinh \left[\frac{(t-s)}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right]}{\sinh \left[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right]^2} e^{\gamma_- (s+\tau)/2} \quad (54)$$

$$C_v(t) = \frac{\eta}{4\pi} \int_0^t d\tau \int_0^t ds K(\tau - s) \frac{\sinh \left[\frac{(t-s)}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right] \sinh \left[\frac{(t-\tau)}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right]}{\sinh \left[\frac{t}{2} \sqrt{\gamma_-^2 - 4\omega_t^2} \right]^2} e^{\gamma_- (s+\tau)/2}. \quad (55)$$

$\mathcal{N}(t)$ is a time dependent coefficient resulting from the fluctuation around the classical path and will be determined imposing the normalization condition to the final reduced density operator. With the propagator (49) we have all the tools needed for studying the time evolution of the reduced density matrix which we do in the next section with a particular example.

EVOLUTION OF A TWO PARTICLE STATE

Now, we want to study the consequences of the effective interaction induced by the reservoir in the time evolution of a system formed by two particles, without any direct interaction between them. Our attention will be focused on the entanglement dynamics of a bipartite system coupled to an oscillator bath and, for clarity, a brief introduction about how to measure entanglement in Gaussian states is necessary.

A Gaussian state can be described using the characteristic function, which in general can be written as

$$\tilde{W}(\mathbf{X}) = \exp \left\{ -\frac{1}{2} \mathbf{X} \boldsymbol{\Lambda} \mathbf{X}^T \right\}, \quad (56)$$

where $\mathbf{X} = (\nu_1, \lambda_1, \nu_2, \lambda_2)$ and $\boldsymbol{\Lambda}$ is the covariance matrix with elements $\Lambda_{ij} = \langle X_i X_j + X_j X_i \rangle / 2$. The characteristic function is related to the density matrix through the transform

$$\tilde{W}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \int_{-\infty}^{\infty} \exp(-i\boldsymbol{\lambda}\mathbf{x}/\hbar) \left\langle \mathbf{x} - \frac{\boldsymbol{\nu}}{2} \right| \hat{\rho} \left| \mathbf{x} + \frac{\boldsymbol{\nu}}{2} \right\rangle d^2x, \quad (57)$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{\nu} = (\nu_1, \nu_2)$ are a sort of momentum and position variables respectively. The covariance matrix contains all the useful information of the Gaussian state and determines it completely. The covariance matrix shall be conveniently written in terms of 2×2 block submatrices as

$$\boldsymbol{\Lambda} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \quad (58)$$

A very common quantifier of entanglement is the logarithmic negativity $E_{\mathcal{N}}$ [14], defined by

$$E_{\mathcal{N}}(\rho) = \max [0, -\ln 2\tilde{\sigma}_-]. \quad (59)$$

where $\tilde{\sigma}_{\pm}$ are the symplectic eigenvalues of the partial transposed density matrix that can be written using the local symplectic invariants $\det \Lambda$, $\det A$, $\det B$, $\det C$ as

$$\tilde{\sigma}_{\pm} = \frac{1}{\sqrt{2}} \left[\tilde{D}_{\Lambda} \pm \sqrt{\tilde{D}_{\Lambda}^2 - 4 \det \Lambda} \right]^{\frac{1}{2}}, \quad (60)$$

where $\tilde{D}_{\Lambda} = \det A + \det B - 2 \det C$. This expression quantifies directly the violation of the PPT (*positive partial transposition*), a necessary and sufficient condition of separability.

We choose as initial state a Gaussian density matrix of two modes characterized by the squeeze parameter z . The initial density matrix written in the variables r , u , v , χ , is

$$\begin{aligned} \tilde{\rho}(r', \chi', u', v', 0) = \\ \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{e^{-2z}}{4\sigma^2} (4r'^2 + \chi'^2) - \frac{e^{2z}}{4\sigma^2} (u'^2 + v'^2/4) \right\}. \end{aligned} \quad (61)$$

The density matrix at $t > 0$ is obtained from the expression (28), where the propagator J and the initial density matrix are given respectively by (49) and (61), with the suitable change of variables. The integrals indicated in (28) can be solved exactly for all the limits mentioned above since the involved variables appear only up to second order. After the evolution of those integrals the covariance matrix is easily computed by the Fourier transform (57) and the results of (59) are presented for various parameter combinations in the figures below.

First, lets us observe the evolution of the logarithmic negativity when the two parts of the system of interest are in the local (short distance) regime. In figure (1) we have the logarithmic negativity as a function of time for low temperatures and three values of the squeeze parameter. The dependence on the initial entanglement is evident in this figure. For a initial separable system ($z = 0$) the environment rapidly induces entanglement and on the average maintains a value higher than that for systems initially entangled in which the decoherence process drastically reduces the logarithmic negativity. The oscillatory behavior is due to the effective interaction potential induced by the environment [1] and it is remarkable that in all the cases plotted in figure (1) there exist a remanent finite entanglement.

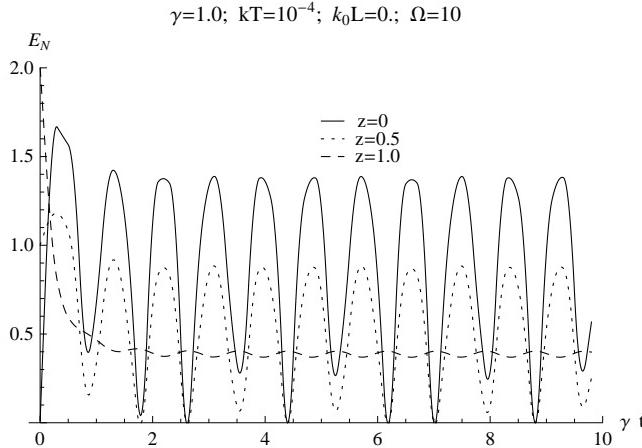


Figure 1: logarithmic negativity for close particles as a function of time, the parameters are specified at the top of the figure.

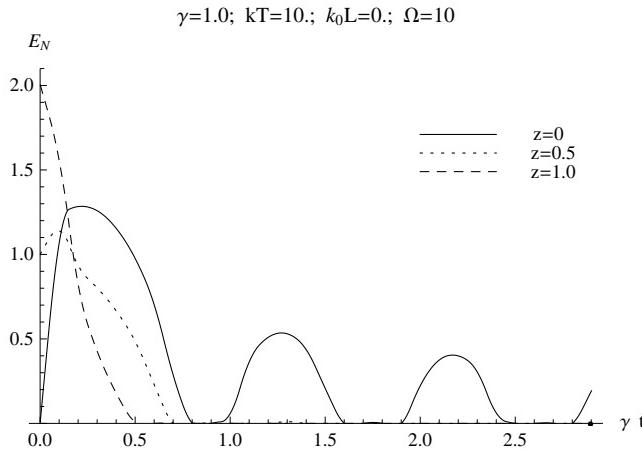


Figure 2: logarithmic negativity for close particles as a function of time. The parameters are specified at the top of the figure .

Figure (2) show us that the remanent entanglement is an effect associated with low temperatures and when the temperature is raised the initial entanglement always go to zero. Again in this case, the initially separable state is the most robust and maintains the induced entanglement for longer times.

When the particles are very close together the most important influence of the reservoir is the effective interaction induced between them and the decoherence effect only becomes important when the temperature rises. In figure (3) the logarithmic negativity is plotted for three values of the dissipative constant γ . The average amount of entanglement is higher for higher values of γ , i.e. stronger interaction with the reservoir. Results similar to this, but following other formalism and always considering a bilinear system-reservoir coupling, have been presented by other autors [15].

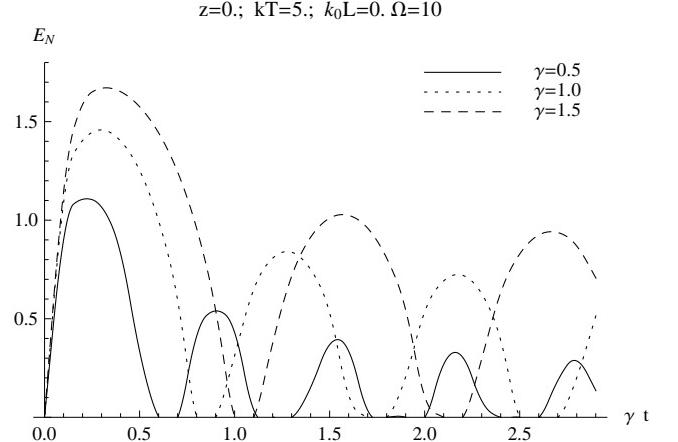


Figure 3: logarithmic negativity for close particles and finite temperature as a function of time. The parameters are specified at the top of the figure .

Up to this point the results showed correspond to bipartite system for which it is possible to define an average distance L between the two parts and this distance is very small when compared with the characteristic length of the reservoir, k_0^{-1} . This approximation assumes that the relative coordinate is completely free from any dissipative influence. In a more realistic situation a dissipative dynamics for the center of mass and relative motions is expected and it is plausible to find a monotonically decreasing remanent entanglement when the distance between the particles is increased. In figure (4) we can observe the influence of the distance between the particles on the evolution of entanglement. It seems there is a competition between the process of entanglement induction and decoherence. In particular, when the distance increases the average entanglement is reduced and the oscillations become damped. In the curve for $k_0L = 2.0$ a delay in establishing entanglement can be seen. Increasing the distance L reduces the induced interaction and increases the delay time. In the limit $k_0L \rightarrow \infty$ the delay time goes to infinity and for finite distances an asymptotically entangled state when the temperature is low enough is observed.

If the initial state is entangled the delay time is a revival time. The initial entanglement is lost by decoherence due to the action of the reservoir and after that the system is again entangled by the induced interaction. This effect is only appreciable at low temperatures. Then, it is possible to tune the induced entanglement process varying the parameters and, in particular, controlling the distance between the subsystems as shown in figure (5).

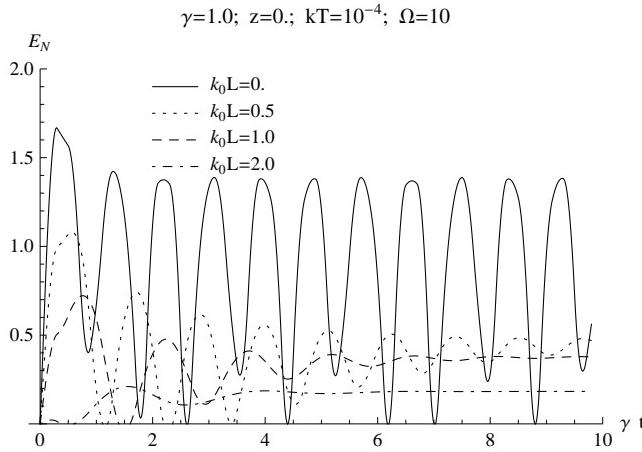


Figure 4: Influence of the distance between the particles on the logarithmic negativity for low temperature. The parameters are specified at the top of the figure.

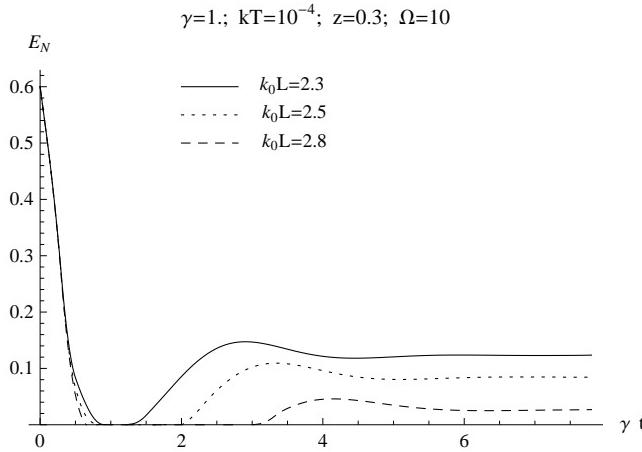


Figure 5: Influence of the distance on the revival of quantum correlations

CONCLUSIONS

In this paper we presented a system-plus-reservoir model with a nonlinear coupling in the system coordinates which, in the adequate limit, allows us to reproduce the phenomenological results known for the quantum dynamics of a Brownian particle. The model was extended for dealing with a system of two uncoupled particles immersed in a common environment. The choice made for the behavior of the bath response function, in order to reproduce the dynamics of the quantum brownian motion for each particle when isolated, naturally generates an effective interaction which depends on the average distance between the particles and the dissipation constant γ , as can be seen from the exact density operator. When the particles are very close together the interaction is basically harmonic and the amplitude of the

induced quantum correlations oscillates without decaying when the temperature is low. In this approximation the dynamics is local and the system preserves the induced entanglement. It is worth noticing that when the initial state is entangled, the system evolution is less sensitive to the induced correlations and more sensitive to the decoherence effects. For interparticle distances comparable with the characteristic length k_0^{-1} of the reservoir other phenomena appear, in particular we have the possibility controlling the time interval between the total lack of initial entanglement and the induced entanglement.

The results presented here may be relevant for a better understanding of the bipartite systems with regard to the theory of quantum information and the extension to deal with many Brownian particles might contribute to the study of quantum correlations of multipartite systems, and of the behavior of complex many particles systems in general.

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